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Kinematic Wave Approach to Hydraulic Jumps with Waves

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Rayleigh's classical approach to hydraulic jumps is discussed and extended to allow for the added presence of finite amplitude disturbance waves. In this paper the decrease of mean flow energy through the jump discontinuity is accounted for by the sudden appearance of enhanced downstream radiating waves. Without appealing to turbulent dissipation, the proposed model, which applies only to weak inviscid bores, relates the mean height, the mean speed, the wave energy density, and the wavenumber fore and aft of the jump through the conservation laws for total mass, total momentum, total energy, and wave crest number. The model, in shallow water, completely describes the undulatory character of the wavy downstream flow, yielding results in agreement with observed features of bores in nature. "Deep water jumps," although somewhat speculative, are also briefly described, with the aim of stimulating further discussion. The relationships between the present hydraulic model and some work of Benjamin and Lighthill and others are also cited for completeness. The basic notions are then extended to "wave discontinuities" in arbitrary continuous media using ideas derived from kinematic wave theory, and a further stability analysis points to the conditions under which the postulated jumps exist.

Introduction

THE transition a physical system undergoes when it jumps from one equilibrium state to another is generally studied by seeking multivalued solutions to the governing nonlinear equations with certain prescribed fluxes assumed. Examples well-known in fluid mechanics include hydraulic jumps in shallow water; gasdynamic shocks in compressible flow; and, possibly, the sudden characteristic increases in core size observed in vortex breakdown. The relationships connecting the conjugate states of a discontinuous flow, when they do exist, and the stability properties of the resulting jumps, of course, depend exactly upon which fluxes are fixed across the jump and upon the dynamical or kinematical model used.

For example, Rayleigh's treatment of hydraulic jumps in shallow water[1] relates the surface elevation h and the flow speed U fore and aft of the discontinuity by conserving mass and momentum; the resulting decrease of energy across the jump, in this classical model, is accounted for by turbulent dissipation. This well-known description applies to strong bores. Experimentally, however, it is found that weak bores have a stationary train of waves behind them: the required loss of mean flow energy need not occur entirely through friction but by the radiation of a stationary wavetrain.[2] This wave radiation, first modeled in the pioneering work of Benjamin and Lighthill,[3] is reconsidered in the present paper, where a new description for hydraulic jumps with finite amplitude waves applicable to weak bores completely absent of any dissipation is given.

The main ideas, developed and discussed in the context of hydraulic jumps in shallow water, are given in the next section where reference to existing analytic models and experiment is also made. The basic approach, however, applies to discontinuous states in arbitrary continuous media, and is generalized using the kinematic wave formalism recently developed by Hayes[4]; a linearized analysis for the stability of the resulting discontinuities with respect to self-induced disturbances, using this general approach, is also carried out, and explicit formulas are obtained for the local amplification or decay rates in terms of spatial gradients in the flow nonuniformity.

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Gravity Waves: an Illustrative Example

The classical treatment of hydraulic jumps in shallow water assumes a mean flow of height h and speed U without superposed waves.[1] Conjugate solutions for U and h, determined by fixing both mass and momentum fluxes, are easily obtained; the directionality of the physical flow is defined by an entropy condition prescribing a decrease in net energy flux on reaching the downstream side, where the loss is attributed to turbulent dissipation. This reasonable hypothesis for strong transitions does not necessarily apply to weaker jumps. Indeed, Favre[2] finds experimentally that weak bores have a stationary train of waves behind them; these waves exhibit no breaking for mean depth ratios less than 1.28. Lemoine,[5] quoting Favre's result, points out that under these circumstances the required loss of energy may occur not by friction, but by radiation through the stationary wavetrain.

This paper considers a simple analytic model for the combined wave and mean flow system. The model completely describes the suddenly enhanced character of the downstream undulatory flow once upstream conditions are prescribed. The presence of waves requires two additional parameters: say, the wave energy density E and the wavenumber K. To connect the parameters h, U, E, and K fore and aft of the discontinuity, four conservation laws are invoked. Since the assumed transition is weak and basically inviscid, we conserve total mass, total momentum, total energy, and the number of wave crests. (The model can be developed assuming, of course, some loss of energy.)

The implicatons of the model are best understood by considering some specific examples. Consider water of finite depth. As discussed, four parameters, namely, E, K, h, and U, are required to describe the physical system. Usually a "mass transport velocity" $\hat{U}$, measuring the nonharmonic part of the complete flow, including the effect of wave-induced changes to the "original" mean flow U without waves, is substituted in place of U. [For a more extensive discussion and the derivation of Eqs. (4-6) see Phillips[6] or Whitham.[7]] Let U_s denote the shock speed and introduce the relative velocity $u = \hat{U} - U_s$. In the shallow water limit characteristic of bore-like flows, Whitham's nonlinear modulation equations, derived for near-linear sinusoidal wavetrains, simplify somewhat because the wavenumber K does not explicitly appear in the conservation laws used. Let m, $\tilde{P}$, and $\tilde{Q}$ be the prescribed fluxes for total mass, total momentum, and total energy; let ρ be the fluid density and g be the acceleration due to gravity, and introduce non-dimensional variables with the definitions

$$\bar{h}=h/\left(\frac{2m^2}{g\rho^2}\right)^{1/3}=h/h_0 \tag{1}$$

$$\bar{u}=u/\left(\frac{gm}{2\rho}\right)^{1/3}=u/u_0 \tag{2}$$

$$\bar{E}=E/\left(\frac{4}{27}\frac{gm^4}{\rho}\right)^{1/3}=E/E_0 \tag{3}$$

The cited conservation laws reduce, respectively, to

$$hu=1 \tag{4}$$

$$hu^2+h^2+E=\frac{\tilde{P}}{\rho h_0 u_0^2}\equiv P>0 \tag{5}$$

$$hu^3+4uh^2+\frac{4}{3}\sqrt{2}h^{1/2}E+\frac{10}{3}Eu=\frac{2\tilde{Q}}{\rho h_0 u_0^3}\equiv Q>0 \tag{6}$$

where "bars" have been dropped for convenience. [When $E=0$ and Eq. (6) is deleted, our formulation reduces to the classical one conserving mass and momentum fluxes only.] From Eqs. (4, 5 and 6), the following ninth-order equation for $H=h^{1/2}$ can be derived:

$$H^9-\frac{1}{4}\sqrt{2}H^6-PH^5+\frac{3}{8}\sqrt{2}QH^4+H^3-\frac{5}{4}\sqrt{2}PH^2+\frac{7}{8}\sqrt{2}=0 \tag{7}$$

For positive P and Q, all positive roots for H require $E\geq 0$ to qualify as physically realistic solutions. Some idea of the jumps typical of Eqs. (4-6) is obtained on comparison with those found through the classical hydraulic model. Denote by "1" and "2" the upstream and downstream states, respectively, and require $h_2>h_1$ in the usual way. For the case $P=2.12$, which corresponds to a Froude number $Fr=3^{1/2}$, where the standard definition $Fr=u/(gh)^{1/2}$ is used, we have $h_2/h_1=2$, $h_1=0.55$, and $h_2=1.10$. The classical mean flow energy flux $Q_m=hu^3+4uh^2=4h+h^{-2}$ is 5.50 upstream and 5.23 downstream, so that the deficit ratio is 5.23/5.50, or 0.95.

In the present model, the loss in mean flow energy is completely accounted for by the presence of radiating waves. For comparison, we set $P=2.12$ and $Q=5.50$. The conjugate depths are now obtained as $h_1=0.55$ and $h_2=1.03$, and calculations show $E_1=0.00$ while $E_2=0.084$. In other words, the waves are practically nonexistent on side 1 but, on side 2, they acquire a sudden visibility. The ratio of downstream to upstream mean energy flux is 0.92, in contrast to 0.95, and the rest goes into waves. The phase velocity decreases across the jump; since the wave frequency, as postulated, is fixed, the wavenumber therefore increases. (Whitham's conservation law for wave crests[7] $K_t+\omega_x=0$, were ω is the wave frequency and x and t are propagation space and time coordinates, reduces to $\omega=$constant in the "steady" $\partial/\partial t=0$ limit; the actual motion is, of course, dynamically unsteady.) Thus, the sudden visibility of the wave is enhanced by both increased wave amplitude and wavenumber, these increases being completely determinable from Eqs. (4-6). Moreover, the classical hydraulic jump behavior for the mean flow is qualitatively reproduced without appeal to turbulent dissipation.

Additional results are obtained by seeking positive E solutions of Eq. (7). A simple way to construct solutions follows by rewriting Eq. (7) in the form

$$P=\frac{3/8\sqrt{2}H^4}{H^5+5/4\sqrt{2}H^2}Q+\frac{H^9-1/4\sqrt{2}H^6+H^3+7/8\sqrt{2}}{H^5+5/4\sqrt{2}H^2} \tag{8}$$

Thus, every value of the parameter H defines a straight line in the P-Q plane. The intersection of two or more lines at some point (P^*,Q^*) indicates two or more values of H at that point. This requires us only to study the slope and P-intercept functions of Eq. (8). For $E\geq 0$, it is necessary that $P\geq P_0=H^4+H^{-2}$. Some typical results are shown in Fig. 1.

Various sets of (H_1,H_2) are considered with $h_2/h_1=2.25$, that is, we assume (H_1,H_2) as a conjugate pair. Each H then determines a straight line in the P-Q plane, and the intersection point (P^*, Q^*) is determined from the solution of two derived linear algebraic equations. Repeating the procedure for different h's generates different solutions, as given in Fig. 1. The results show how every point along the solid portion of the line determines two conjugate solutions: the jump in h from h_1 to h_2 is always accompained by an increase in E and a decrease in Q_m, agreeing qualitatively with Rayleigh's hydraulic model. Each point along the solid curve posesses two roots with positive E solutions; the dashed portion yields one negative root for E, however, so that discontinuous solutions do not exist. It follows that hydraulic jumps in shallow water, under the assumed model, are possible only for sufficiently large P's or Q's.

General Approach and Stability Analysis

The basic ideas underlying the foregoing analysis are quite general and apply to hydraulic discontinuities in arbitrary continuous media. The nonlinear wave and mean flow interactions that inevitably arise in problems such as those considered in this paper are, fortunately, amenable to simple mathematical representation using recently developed ideas from kinematic wave theory; and, in this section, we redevelop our ideas more generally. In what follows, we adopt Haye's[4] Hamiltonian approach to Whitham's[7] average Lagrangian method and assume some familiarity with these techniques.

The problems under consideration are described more generally by introducing two triads, namely, (P,γ,β) and (A,ω,K), an explicit dependence on the nonuniformity $\lambda(x,t)$, and an average Lagrangian $\mathcal{L}$ in the form

$$\mathcal{L}=A\omega+P\gamma-H(K,A,\beta,P,\lambda) \tag{9}$$

following Hayes. The first triad characterizes mean properties; in the context of the foregoing example, P and β would correspond, respectively, to the mean height and the mean speed, while $\gamma=\gamma(\beta,P,A,K)$, determined from the auxiliary condition $\mathcal{L}_P=0$ or $\gamma=H_P$, would be a Bernoulli-type "dispersion relation" for the pressure. The second triad (A,ω,K) characterizes the coupled wave flow; A, ω, and K refer to, respectively, the wave action density, the wave frequency, and the wavenumber, where the dispersion relation obtains from $\mathcal{L}_A=0$ or $\omega=H_A$.

Four remaining equations are needed to connect the six unknowns. As discussed, we conserve wave crest number, total mass, total energy, and total momentum. In a coordinate system moving with a shock of speed $U_s(t)$, where t is time, we have

$$H_A(K,A,\beta,P,\lambda)-U_sK=\text{constant} \tag{10}$$

$$H_\beta-U_sP=\text{constant} \tag{11}$$

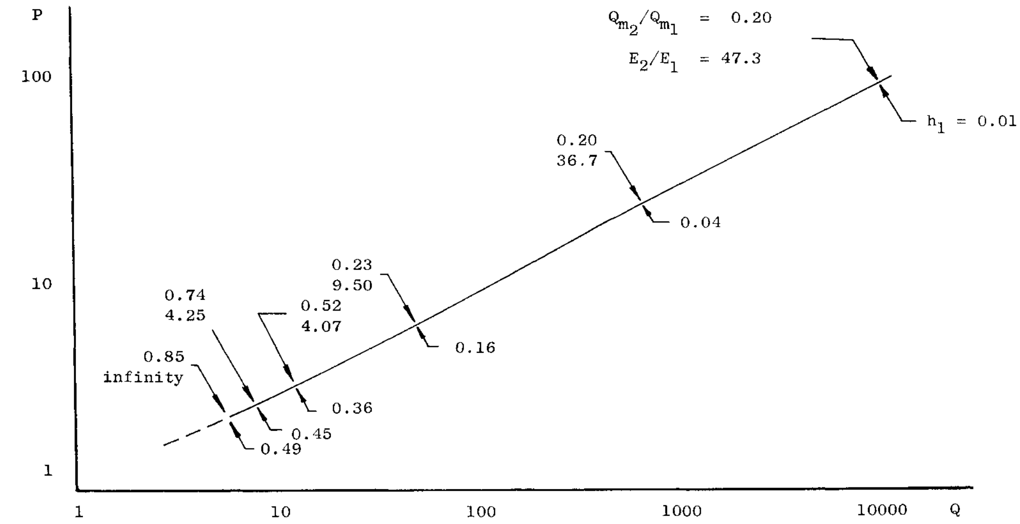

Fig. 1 Shallow water gravity waves: conjugate solutions for $h_2/h_1 = 2.25$.

$$H_A H_K + H_P H_\beta - U_s H = \text{constant} \tag{12}$$

$$AH_A + PH_P - H + KH_K + \beta H_\beta - U_s(KA + \beta P) = \text{constant} \tag{13}$$

We now assume a basic steady solution and displace the stationary shock with a small amount $\Xi(t)$. Expansion of all flow parameters about the steady solution corresponding to $\Xi = 0$, noting that the shock speed $U_s(t) \cong \dot{\Xi}(t)$ to $0(\Xi)$, where dots refer to time derivatives, and that, for example, $A \cong A_0 + A_0'\Xi$ and

$$F(A) \cong F(A_0) + F_A(A_0)A_0'\Xi$$

for any function $F(A)$, yields

$$H_{A_{1,0}} = H_{A_{2,0}} \tag{14}$$

$$H_{\beta_{1,0}} = H_{\beta_{2,0}} \tag{15}$$

$$H_{A_{1,0}} H_{K_{1,0}} + H_{P_{1,0}} H_{\beta_{1,0}} = H_{A_{2,0}} H_{K_{2,0}} + H_{P_{2,0}} H_{\beta_{2,0}} \tag{16}$$

$$\begin{aligned} &A_{1,0}H_{A_{1,0}} + P_{1,0}H_{P_{1,0}} - H_{1,0} + K_{1,0}H_{K_{1,0}} + \beta_{1,0}H_{\beta_{1,0}} \\ &= A_{2,0}H_{A_{2,0}} + P_{2,0}H_{P_{2,0}} - H_{2,0} + K_{2,0}H_{K_{2,0}} + \beta_{2,0}H_{\beta_{2,0}} \end{aligned} \tag{17}$$

to 0(1). Subscripts 0 here refer to the undisturbed state; Eqs. (14-17) correspond to the jump conditions described in the previous section, but now they are written in a general form applicable to a wider variety of problems. The $0(\Xi)$ hierarchy of equations leads to, respectively,

$$\dot{\Xi} = \frac{(\ )_2 - (\ )_1}{K_{2,0} - K_{1,0}}\,\Xi \tag{18a}$$

$$(\ ) = H_{AK}K' + H_{AA}A' + H_{A\beta}\beta' + H_{AP}P' + H_{A\lambda}\lambda' \tag{18b}$$

$$\dot{\Xi} = \frac{(\ )_2 - (\ )_1}{P_{2,0} - P_{1,0}}\,\Xi \tag{19a}$$

$$(\ ) = H_{\beta K}K' + H_{\beta A}A' + H_{\beta\beta}\beta' + H_{\beta P}P' + H_{\beta\lambda}\lambda' \tag{19b}$$

$$\dot{\Xi} = \frac{(\ )_2 - (\ )_1}{H_{2,0} - H_{1,0}}\,\Xi \tag{20a}$$

$$\begin{aligned} (\ ) = {} & H_A(H_{KK}K' + H_{KA}A' + H_{K\beta}\beta' + H_{KP}P' + H_{K\lambda}\lambda') \\ & + H_K(H_{AK}K' + H_{AA}A' + H_{A\beta}\beta' + H_{AP}P' + H_{A\lambda}\lambda') \\ & + H_P(H_{\beta K}K' + H_{\beta A}A' + H_{\beta\beta}\beta' + H_{\beta P}P' + H_{\beta\lambda}\lambda') \\ & + H_\beta(H_{PK}K' + H_{PA}A' + H_{P\beta}\beta' + H_{PP}P' + H_{P\lambda}\lambda') \end{aligned} \tag{20b}$$

$$\dot{\Xi} = \frac{(\ )_2 - (\ )_1}{(K_{2,0}A_{2,0} + \beta_{2,0}P_{2,0}) - (K_{1,0}A_{1,0} + \beta_{1,0}P_{1,0})}\,\Xi \tag{21a}$$

$$\begin{aligned} (\ ) = {} & A(H_{AK}K' + H_{AA}A' + H_{A\beta}\beta' + H_{AP}P' + H_{A\lambda}\lambda') \\ & + P(H_{PK}K' + H_{PA}A' + H_{P\beta}\beta' + H_{PP}P' + H_{P\lambda}\lambda') \\ & + K(H_{KK}K' + H_{KA}A' + H_{K\beta}\beta' + H_{KP}P' + H_{K\lambda}\lambda') \\ & + \beta(H_{\beta K}K' + H_{\beta A}A' + H_{\beta\beta}\beta' + H_{\beta P}P' + H_{\beta\lambda}\lambda') \\ & - H_\lambda\lambda' \end{aligned} \tag{21b}$$

The bracketed quantities in Eqs. (18-21) can be simplified using the wave action equation $\partial H_K/\partial x = 0$, the wave conservation law $\partial H_A/\partial x = 0$, the mass flux law $\partial H_\beta/\partial x = 0$, and the consistency conditions $\partial H_P/\partial x = 0$ and $\partial\beta/\partial t + \partial\gamma/\partial x = 0$. On substitution, the amplification rates in the first three equations vanish identically, implying neutral stability. However, Eq. (21) is nontrivial and leads to

$$\dot{\Xi} = \frac{-H_{\lambda_{2,0}}\lambda_{2,0}' + H_{\lambda_{1,0}}\lambda_{1,0}'}{(K_{2,0}A_{2,0} + \beta_{2,0}P_{2,0}) - (K_{1,0}A_{1,0} + \beta_{1,0}P_{1,0})}\,\Xi \tag{22}$$

Thus, when the inhomogeneity λ' vanishes, the shock is neutrally stable. (Our stability criterion does not include high-order nonlinear effects or dissipation, however.) Eqs. (14-17 and 22) are the main contributions of this section, and examples of their application are given by Chin.[8]

Before concluding this section, it is instructive to apply our stability results to hydraulic jumps in water of finite depth. The following expansion for H, obtained by assuming a near-linear Stokes series, is found in Hayes,[4]

$$H = \tfrac{1}{2}\beta^2 P + \tfrac{1}{2} g P^2 + A\beta K + A\omega^0 (K,P) + \tfrac{1}{2} K^3 D A^2 + 0(A^3)$$

$$\omega^0 = (gKT)^{1/2} \qquad T = \tanh KP$$

$$D = 9/8T - 5/4T^{-1} - 9/8T^{-3} \tag{23}$$

where, again, g is the acceleration due to gravity. In the general case, both the mean speed β and the mean height P may vary with changes in x. We assume a flat bottom, however, with streamwise inhomogeneity appearing only through the appearance of a variable $\beta(x)$ induced, say, by some type of side inflow. (Hydraulic jumps are usually formed on variable currents and their stability and observability directly depends on the type of nonuniformity.) Setting $\lambda(x) = \beta(x)$ shows that $H_\lambda \lambda' = (KA + \beta P)\beta'$. If $\beta'(x)$ is continuous through the jump, Eq. (22) yields.

$$\dot{\Xi} = -\beta'(x_{\text{shock}})\Xi \tag{24}$$

Thus the shock is unstable in a region of decelerating flow and stable in a region of accelerating flow; instability arises because the infinitesimal disturbances assumed here continually consume the energy of the basic wave and mean flow system. Incidentally, this criterion is exactly the same that obtains for the stability of gasdynamic shocks in slowly varying channels.

Discussion and Concluding Remarks

This paper generalizes the notion of a hydraulic jump by introducing the idea of a combined nonlinearly coupled wave and mean flow system. The added presence of a wave requires two additional parameters in the overall description, E and K, as presented here and, therefore, two additional conservation laws; conservation of total mass, total momentum, total energy, and wave crests is assumed. The use of a basic sinusoidal wave solution obtained from a near-linear Stokes type expansion to account for losses in mean flow energy implies a weak nonbreaking transition consistent with the "conserving" jump conditions chosen. The general theory of wave discontinuities and their stability with respect to self-induced disturbances is developed using the formalism of kinematic wave theory, and the method is applied in detail to hydraulic jumps in shallow water. The approach, as noted, yields results consistent with observed features of bores in nature.

Simple algebraic formulas for end results amenable to hand calculation for the example considered, unfortunately, are not possible except in the more elementary classical hydraulic jump limit without waves. However, detailed numerical results are easily obtained from Eqs. (4-7). The "inverse" problem discussed in the text just following Eq. (8), for example, describes one way of generating families of conjugate solutions; in this method, the mean depth ratio h_2/h_1 is fixed and P and Q are determined in the process. Often the fluxes P and Q are to be specified beforehand and multivalued solutions are sought. The solution of this "direct" problem calls for the following simple recipe. Real positive solutions of Eq. (7) for H are first obtained, these corresponding to physically realistic values provided that E, on the basis of Eqs. (4) and (5), say, is non-negative; the streamwise speed u, of course, follows directly from Eq. (4), while the wavenumber is solved from the requirement that the wave frequency takes on the same prescribed constant value on both sides of the discontinuity. (In this sense the problems considered are "steady," meaning that $\partial K/\partial t + \partial\omega/\partial x = 0$, with $\partial/\partial t = 0$, yields ω = constant; the actual dynamical problem, naturally, is an unsteady one.)

An intriguing question arises in the deep water limit: Do discontinuous solutions exist? If, as assumed, U_m is uniform over depth, we must disallow discontinuities in the mean speed because they would imply unrealistic infinite changes in mean flow energy. Thus, if jumps in E and K cannot be accompanied by jumps in U_m and h, the wave and mean fields must, in this limit, dynamically uncouple. (This decoupling was first indicated by Whitham[7] in a different context.) On this basis, conjugate solutions for E and K are separately determined, quite plausibly, by fixing the flux of wave momentum and by fixing the wave frequency, if dissipation is assumed to act in the detailed description.

Discontinuous solutions obtained in this manner, dependent on the parameter U_m, are discussed by the present author in Ref. 8. When conjugate values do exist, they are characterized by jumps across which E and K increase, while net energy flux decreases; the stability of these "deep water jumps," after some analysis, also follows Eq. (24). The deep water problem, of course, is a nontrivial limit of the finite depth formulation. While the formal results cited here are interesting, there is no current experimental evidence supporting their existence; but they may be relevant to wave breaking and other catastrophic transitions in the ocean. For further discussion, the reader is referred to Ref. 8, where the possibility of re-establishing the wave and mean flow coupling in deep water through an assumed dependence of the mean speed U_m on the vertical coordinate y is also considered.

The study of discontinuous solutions has gained much recent attention, for example, Benjamin's[9] theory of conjugate flows in vortex breakdown, which develops further the early ideas of Ref. 3, or the alternative approach of Landahl and Widnall,[10] also for vortex breakdown. These theories, as in the present paper, in effect, model rapid transitions by idealizing the basic flow configuration using abrupt discontinuous changes in state. The most interesting study relevant to the present analysis, of course, is the pioneering study of Benjamin and Lighthill[3] for hydraulic jumps in water of finite depth. This original work, also motivated by the experimental results of Favre,[2] was carried out to complement some theoretical estimates of Lemoine[5] on the properties of radiating sinusoidal waves of infinitesimal amplitude. (Ref. 3 points out some numerical errors of Lemoine, however.) The Benjamin and Lighthill model, in contrast, was pursued to establish the extent to which the assumption of a radiating cnoidal wave holds, and numerical results are given in that paper; these authors show that such a wavetrain, present in all but very strong bores, is capable of absorbing almost the whole energy which, according to classical theory, is liberated at the bore—noting, however, that some minute residual dissipation of energy appears to be necessary. The model considered in the present paper considers finite amplitude waves of sinusoidal form, and the nonlinear wave and mean flow coupling is furnished through the kinematic wave model described in Whitham's modulation equations [see Eqs. (4-6)]. The results of the foregoing analysis show that hydraulic solutions of the type considered need not exist for all flux parameters, but the qualitative effect of wave radiation appearing in the downstream flow is somewhat similar to that of Benjamin and Lighthill. The present model can, of course, be slightly modified to allow some energy dissipation through the jump, by reworking the total energy balance; work along these lines, which has not yet been pursued, would permit a meaningful comparison with Ref. 3. For completeness, we note that the approach taken here on shock stability and mean and perturbation flow interaction follows the well-known ideas of Kantrowitz[11] and Burgers[12] in gasdynamics.

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